variables this result is not generally true.* The following considerations furnish a simple proof of the theorem stated above when $m_1, m_2, \ldots, m_\alpha$ are numbers.

Let $m$ be any number which is divisible by $(m_1, m_2, \ldots, m_\alpha)$ and consider the two systems

$$(m, m_1, m_2, \ldots, m_\alpha) \text{ and } (m_1, m_2, \ldots, m_\alpha).$$

Since every element of each of these systems is divisible by the other system we have $(m, m_1, m_2, \ldots, m_\alpha) \backsim (m_1, m_2, \ldots, m_\alpha)$.

That is, *if any multiple of the elements of a modular system be added to its elements the resulting system is equivalent to the original system.*

In particular, the two systems $(m_1+tm_2, m_1, m_2, \ldots, m_\alpha)$ and $(m_1, m_2, \ldots, m_\alpha)$ are equivalent for all values of $t$. The former of these is clearly equivalent to $(m_1+tm_2, m_2, \ldots, m_\alpha)$ since

$$m_1=(m_1+tm_2)-tm_2.$$

Hence it follows that $(m_1, m_2, \ldots, m_\alpha) \backsim (m_1+tm_2, m_2, \ldots, m_\alpha)$.

That is, *any modular system is equivalent to the one obtained by increasing or decreasing one of its elements by any multiple of any other element of the system.*

Suppose that a given modular system involves at least two elements which differ from 0. The preceding theorem enables us to find an equivalent system in which one of the elements is reduced by an integer. Hence every modular system is equivalent to a system in which all the elements except one are zeros. Since two systems which differ only with respect to 0 elements are equivalent, it follows that every modular system is equivalent to a system containing only one element, as was stated above. That is, it is always possible to find a number $d$ such that

$$(m_1, m_2, \ldots, m_\alpha) \backsim (d).$$

It remains only to find $d$. Since each of the elements $m_1, m_2, \ldots, m_\alpha$ is divisible by $d$ it follows that $d$ is a common divisor of the elements of $(m_1, m_2, \ldots, m_\alpha)$. The other condition imposed by this equivalence, viz.,

$$d=c_1m_1+c_2m_2+\ldots+c_\alpha m_\alpha,$$

requires that $d$ be divisible by the greatest common divisor of $m_1, m_2, \ldots, m_\alpha$. Hence $d$ must be the greatest common divisor of the elements of

$$(m_1, m_2, \ldots, m_\alpha).$$

We are now in position to see more clearly why the two systems

$$(20, 35) \quad (15, 25, 10)$$

*All the modular systems considered in this note are supposed to have numbers for their elements.

are equivalent, as each of them is equivalent to (5).

Suppose that the greatest common divisor of the elements of the system $(m_1, m_2, \ldots, m_a)$ is unity. From

$$(m_1, m_2, \ldots, m_a) \backsim (1)$$

it follows that it is possible to find $a$ numbers $c_1, c_2, \ldots, c_a$ such that

$$c_1 m_1 + c_2 m_2 + \ldots + c_a m_a = 1.$$

In particular, we have the fundamental theorem that it is possible to find two numbers $x, y$ such that

$$m_1 x + m_2 y = 1,$$

whenever $m_1$ and $m_2$ are prime to each other.

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## GENERALIZATION OF A FUNDAMERTAL THEOREM IN THE GEOMETRY OF THE TRIANGLE.

By PROF. M. W. HASKELL.

The theorem in question is of fundamental importance in the geometry of the triangle,* and may be stated as follows:

*If $A'$, $B'$, $C'$ be points chosen at will on the sides $BC$, $CA$, $AB$ of any triangle $ABC$, the circles $AB'C'$, $BC'A'$, $CA'B'$ pass through one and the same point $O$.*

In a communication presented to the Chicago Section of the American Mathematical Society January 2, 1902, I extended this theorem to the tetrahedron in the following form:

*Let $F, G, H, P, Q, R$ be any points on the edges $AD$, $BD$, $CD$, $BC$, $CA$, $AB$, respectively, of any tetrahedron; the four spheres $AFQR$, $BGRP$, $CHPQ$, $DFGH$ pass through one and the same point $O$.*

The theorem is, however, capable of generalization to space of any number of dimensions without any difficulty. I will therefore state and prove it at once for space of $n$ dimensions,—understanding by a spherical space of three dimensions, $S_3$, a space every section of which by a flat space of three dimensions, $R_3$, is an ordinary sphere, and in general by a spherical $S_{n-1}$ a space every section of which by a flat $R_{n-1}$ is a spherical $S_{n-2}$. A spherical $S_{n-1}$ will evidently be determined by $n+1$ points of which never more than two lie on the same line nor more than three in the same plane, etc. The general theorem may then be stated in the following words:

*McClelland, *Geometry of the Circle*, page 40; see also Rouche et de Comberousse, *Traite de Geometrie*, 7th edition, Vol. I, page 486.

*Let $A_1$, $A_2$, $A_3$....$A_{n+1}$ be the vertices of an $(n+1)$-point in a flat space of $n$ dimensions, and select at will on each edge $A_iA_k$ of this $(n+1)$-point a point $A_{ik}$. The $n+1$ spherical $S_{n-1}$ determined by the groups of points such as $[A_i; A_{i1}, A_{i2}, \ldots A_{i,n+1}]$ will all pass through one and the same point $O$.*

We shall use barycentric coördinates $a'$, $a''$, ....$a^{[n+1]}$ with reference to the given $n$-point, so that

$$\Sigma a^{[i]} = a' + a'' + \ldots + a^{[n+1]} = K.$$

These coördinates are the generalization of triangular coördinates in the plane, in which case $K$ is the area of the triangle of reference, and of tetrahedral coördinates in space of three dimensions, where $K$ is the volume of the fundamental tetrahedron. For the problem in hand it is important to regard the absolute values of these coördinates, and not, as in projective geometry, merely their ratios.

The equation of the region at infinity, a flat $R_{n-1}$, is then

$$\Sigma a^{[i]} = 0;$$

and, if we denote by $a_{ik}$ the length of the edge $A_iA_k$, the equation of the spherical $S_{n-1}$ circumscribing the fundamental $(n+1)$-point $A_1A_2\ldots A_{n+1}$ will be

$$\Omega = \sum_{i=1}^{n} \sum_{k=i+1}^{n+1} a^2_{ik} a^{[i]} a^{[k]} = 0,$$

while the equation of *any* spherical $S_{n-1}$ will be of the form

$$\Sigma \lambda^{[i]} a^{[i]} . \Sigma a^{[i]} - \Omega = 0,$$

where the $\lambda^{[i]}$ are constant coefficients.

Now the coördinates of any vertex $A_i$ of the fundamental $n$-point are of course all zero except $a_i^{[i]}$, which is equal to $K$; and the coördinates of any of the intermediate points $A_{ik}$ are all zero except two, $a_{ik}^{[i]}$ and $a_{ik}^{[k]}$, whose sum is equal to $K$. If for brevity we write

$$\omega_i = \sum_{k=1}^{n+1} a_{ik}^2 a_{ik}^{[i]} a^{[k]}, \quad [k \gtrless i]$$

the equation of the spherical $S_{n-1}$ through the points $[A_i; A_{i1}, A_{i2}, \ldots A_{i,n+1}]$ is readily found to be

$$\omega_i \Sigma a^{[i]} - K\Omega = 0.$$

We see however that

$$K\Omega = \Sigma \omega_i a^{[i]}$$

identically, and hence that *every one of the spherical* $S_{n-1}$ *in question passes through the point* $O$ *for which*

$$\omega_1 = \omega_2 = \omega_3 = \ldots = \omega_{n+1},$$

and the theorem is proved.

We proceed to the discussion of some special cases.

I. If the points $A_{ik}$ are the middle points of the edges, it is evident that

$$\omega_i = \frac{K}{2} \sum_{k=1}^{n+1} a_{ik}{}^2 a^{[k]} = \frac{K}{2} \frac{\partial \Omega}{\partial u^{[i]}},$$

and the point $O$ is the center of the spherical $S_{n-1}$ circumscribed about the fundamental $n$-point.

II. If the flat spaces

$$\omega_1 = 0,\ \omega_2 = 0,\ \ldots \omega_{n+1} = 0$$

have a point in common, the point $O$ will be that point and it will then lie on the circumscribing $S_{n-1}$.

Now this will be the case, for example, if, when $n$ is *even*, the points $A_{ik}$ are the intersections with the edges of a flat $R_{n-1}$. For, let the equation of this $R_{n-1}$ be

$$\lambda_1 a' + \lambda_2 a'' + \ldots + \lambda_{n+1} a^{[n+1]} = 0.$$

The coördinates of $A_{ik}$ will then be

$$a_{ik}{}^{[i]} = \frac{\lambda_k . K}{\lambda_k - \lambda_i},\ a_{ik}{}^{[k]} = \frac{\lambda_i . K}{\lambda_i - \lambda_k}$$

and the resultant of the equations $\omega_1 = 0$ will be the determinant

$$\begin{vmatrix}
0 & \frac{a_{12}{}^2 \lambda_2 K}{\lambda_2 - \lambda_1} & \frac{a_{13}{}^2 \lambda_3 K}{\lambda_3 - \lambda_1} & \ldots\ldots & \frac{a^2_{1,n+1} \lambda_{n+1} K}{\lambda_{n+1} - \lambda_1} \\
\frac{a_{21}{}^2 \lambda_1 K}{\lambda_1 - \lambda_2} & 0 & \frac{a_{23}{}^2 \lambda_3 K}{\lambda_3 - \lambda_2} & \ldots\ldots & \frac{a^2_{2,n+1} \lambda_{n+1} K}{\lambda_{n+1} - \lambda_2} \\
\frac{a_{31}{}^2 \lambda_1 K}{\lambda_1 - \lambda_3} & \frac{a_{32}{}^2 \lambda_2 K}{\lambda_2 - \lambda_3} & 0 & \ldots\ldots & \frac{a^2_{3,n+1} \lambda_{n+1} K}{\lambda_{n+1} - \lambda_3} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
\frac{a^2_{n+1,1} \lambda_1 K}{\lambda_1 - \lambda_{n+1}} & \frac{a^2_{n-1,2} \lambda_2 K}{\lambda_2 - \lambda_{n+1}} & \frac{a^2_{n-1,3} \lambda_3 K}{\lambda_3 - \lambda_{n+1}} & \ldots\ldots & 0
\end{vmatrix}$$

If we now divide the columns of this determinant respectively by $\lambda_1$, $\lambda_2$, $\lambda_3$, $\ldots\lambda_{n+1}$, the quotient is a zero-axial skew determinant, of *odd* order when $n$ is *even* and therefore vanishing. In this case, then, the flat spaces $\omega_1=0$, $\omega_2=0$, $\omega_3=0$, $\ldots\omega_{n+1}=0$ have a point in common; this will be the point $O$ and it lies on the circumscribing $S_{n-1}$. In particular, for the case of a plane triangle ($n=2$), if the points $A_{23}$, $A_{31}$, $A_{12}$ are collinear, the point $O$ lies on the circumscribing circle,—a known theorem.

If, however, $n$ is odd, the above determinant does not vanish. In this case the point $O$ is a point of the intersecting $R_{n-1}$. For, since $O$ is determined by

$$\omega_1=\omega_2=\omega_3=\ldots=\omega_{n+1},$$

we may write

$$\omega_1-\rho=0,\ \omega_2-\rho=0,\ \ldots\omega_{n+1}-\rho=0,$$

and the resultant of these $n+1$ equations and of the equation of the given $R_{n-1}$ is the determinant

$$\begin{vmatrix} 0 & \frac{a_{12}^2\lambda_2 K}{\lambda_2-\lambda_1} & \frac{a_{13}^2\lambda_3 K}{\lambda_3-\lambda_1} & \cdots & \frac{a^2_{1,n+1}\lambda_{n+1}K}{\lambda_{n+1}-\lambda_1} & -1 \\ \frac{a_{21}^2\lambda_1 K}{\lambda_1-\lambda_2} & 0 & \frac{a_{23}^2\lambda_3 K}{\lambda_3-\lambda_2} & \cdots & \frac{a^2_{2,n+1}\lambda_{n+1}K}{\lambda_{n+1}-\lambda_2} & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{a^2_{n+1,1}\lambda_1 K}{\lambda_1-\lambda_{n+1}} & \frac{a^2_{n+1,2}\lambda_2 K}{\lambda_2-\lambda_{n+1}} & \frac{a^2_{n+1,3}\lambda_3 K}{\lambda_3-\lambda_{n+1}} & \cdots & 0 & -1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \cdots & \lambda_{n+1} & 0 \end{vmatrix}$$

and, if we divide the first $n+1$ columns of this determinant by $\lambda_1$, $\lambda_2$, $\lambda_3$, $\ldots\lambda_{n+1}$, respectively, the quotient is a zero-axial skew determinant, which is of odd order if $n$ is *odd*, and will then vanish.

In particular, for the case of a tetrahedron ($n=3$), if the points $A_{ik}$ are coplanar, the point $O$ lies on the same plane, and we have a correspondence between points and planes, in which to every plane is coördinated a point lying in that plane. The relation is not, however, uniquely reversible, as in the case of the ordinary null-system, for to every point $O$ correspond *six* planes.

Finally, if we consider the $S_{n-1}$ in sets of $n$, each such set will have a second point of intersection in addition to the point $O$. These points will be situated in the respective faces of the $n$-point,—meaning by faces the flat $R_{n-1}$ determined by the vertices taken $n$ at a time. It is evident that, if $n$ is odd, these points will lie on the circumscribing $S_{n-1}$; while if $n$ is even, they will lie on the the $R_{n-1}$ whose intersections with the edges determine the points $A_{ik}$.

ROME, ITALY, *December 15, 1902.*

# INTEGRATION AS A SUMMATION.

By PROF. GEORGE R. DEAN, Rolla, Mo.

The following method of presenting the matter seems worthy of a place in elementary text-books, but I have not yet seen it in print.

Let the integral $(a, a+h)$ be divided into $n$ equal parts and find the limit, as $n$ is increased without limit, of the expression

$$f(a).\frac{h}{n}+f(a+\frac{h}{n}).\frac{h}{n}+f(a+\frac{2h}{n}).\frac{h}{n}+\ldots+f(a+\frac{(n-1)h}{n})\frac{h}{n}+f(a+h)\frac{h}{n}.$$

Supposing $f(x)$ capable of development by Taylor's Formula, we have

$$f(a+\frac{h}{n})=f(a)+f'(a).\frac{h}{n}+\frac{f''(a)}{2!}\frac{h^2}{n^2}+\ldots+\frac{f^r(a)}{r!}\frac{h^r}{n^r}+\ldots$$

$$f(a+\frac{2h}{n})=f(a)+f'(a)\frac{2h}{n}+\frac{f''(a)}{2!}\frac{2^2h^2}{n^2}+\ldots+\frac{f^r(a)}{r!}\frac{2^rh^r}{n^r}+\ldots$$

. . . . . .

$$f(a+\frac{rh}{n})=f(a)+f'(a)\frac{rh}{n}+\frac{f''(a)}{2!}\frac{r^2h^2}{n^2}+\ldots+\frac{f^r(a)}{r!}\frac{r^rh^r}{n^r}+\ldots$$

Then $\sum_{r=0}^{r=n} f(a+\frac{rh}{n})\frac{h}{n}=hf(a)+\frac{h^2}{n^2}f'(a)\sum_{r=0}^{r=n} r+\frac{h^3}{n^3}f''(a)\sum_{r=0}^{r=n} r^2+\ldots$

$$+\frac{h^{p+1}}{n^{p+1}}f^p(a)\sum_{r=0}^{r=n} r^p+\ldots$$

By Chrystal's *Algebra*, t. I, p. 487, $\sum_{r=0}^{r=n} r^p=\frac{1}{p+1}n^{p+1}+q_1n^p+\ldots.$

$$L_{n=\infty}\frac{h^{p+1}}{n^{p+1}}\sum_{r=0}^{r=n} r^p=\frac{h^{p+1}}{p+1}.$$

Then $L_{n=\infty}\sum_{r=0}^{r=n} f(a+\frac{rh}{n})\frac{h}{n}=hf(a)+\frac{h^2}{2!}f'(a)+\frac{h^3}{3!}f''(a)+\ldots$

If $f(a)$ is the derivative of $\varphi(a)$, we have, by Taylor's Theorem,

$$\varphi(a+h)-\varphi(a)=hf(a)+\frac{h^2}{2!}f'(a)+\frac{h^3}{3!}f''(a)+\ldots$$

$$\therefore\ \varphi(a+h)-\varphi(a)=\underset{n=\infty}{L}\sum_{r=0}^{r=n}f(a+\frac{rh}{n})\frac{h}{n}.$$

Hence $\int_a^{a+h}f(x)dx=\varphi(a+h)-\varphi(a)$, where $\varphi(x)$ is the anti-derivative of $f(x)$.

## ON THE CHINESE ORIGIN OF THE SYMBOL FOR ZERO.

By PROFESSOR FLORIAN CAJORI.

I have just received a letter from Mr. Y. Mikami, of Tokyo, Japan, containing information which (if confirmed by more extended research) is of great interest and importance. The letter is dated December 15, 1902. From it I quote the following:

"I have found very important relations between the mathematics of India and of China. Arabian numerals seem to be of Chinese origin. The abacus, used by the Chinese from time immemorial, probably afforded the principle of position. In China the use of the symbol 0 for zero seems to have been very old. I desire to study the history of the Chinese mathematics from this point of view, if only I can secure sufficient materials, which is, however, very difficult. Chinese works are not [difficult] to understand for us Japanese, because we use the same letters."

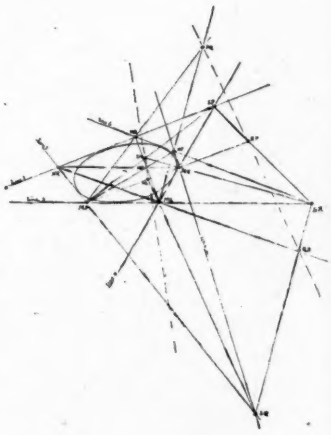
Until recently the symbol for zero and the principle of local value in our notation of numbers were supposed to be of Hindu origin. A few years ago our attention was called to the early work of the Japanese, and now the priority appears to be passing to the Chinese.

COLORADO COLLEGE, COLORADO SPRINGS, *January 3, 1903.*

# THE DERIVATION OF THE BRIANCHON CONFIGURATION FROM TWO SPATIAL POINT-TRIADS.

By ARCHIBALD HENDERSON, Ph. D., Associate Professor of Mathematics, University of North Carolina, Chapel Hill, N. C.

Cayley* has considered the question of deriving the Pascalian configuration, by projection, from a pair of trihedrals. Denote the three planes of one trihedral by $a_1$, $a_2$, $a_3$; of the other by $b_1$, $b_2$, $b_3$. Considering the nine lines $a_i b_j \begin{Bmatrix} i=1,\ 2,\ 3 \\ j=1,\ 2,\ 3 \end{Bmatrix}$ and taking them in a particular way in six sets of three each, we may pass hyperboloids through each set of three lines. These hyperboloids intersect in four points $O_1$, $O_2$, $O_3$, $O_4$ and if we project the solid figure of the two trihedrals from any one of these four points upon an arbitrary plane, the resulting figure is the Pascalian configuration.



This theorem of Cayley's, in connection with certain considerations concerning cubic surfaces, led to the present investigation. Four-point coördinates are used in this paper and a word of explanation is perhaps not amiss. An equation of the form

$$u_1x+u_2y+u_3z+u_4w=0$$

is the equation of a *point*, the coördinates being the variables $u_1$, $u_2$, $u_3$, $u_4$, which are the perpendiculars from the four points $A$, $B$, $C$, $D$ of the fundamental tetrahedron $ABOD$ upon any plane passing through the point in question. If we have two points given by their equations

$$\xi=0,\ \xi'=0,$$

then the equation

$$\xi+\kappa\xi'=0 \quad (\kappa \text{ constant})$$

represents some point on the line joining the two given points. These preliminary remarks should suffice to make the succeeding discussion readily comprehensible.

Consider two point-triads in space, the one triad consisting of the points designated $L$, $M$, $N$, the other consisting of the points designated $P$, $Q$, $R$. In the system of four-point coördinates chosen above, where $u_1$, $u_2$, $u_3$, $u_4$ are current coördinates, the equations of the points are chosen as follows:

*Collected Mathematical Papers, Vol. VI, pp. 129-134.

$$L : u_4=0$$
$$M : mu_1+lu_2+lmnu_3+(mn-l)(nl-m)(lmn-1)u_4=0$$
$$N : nlu_1+mnu_2+u_3-(mn-l)(nl-m)(lmn-1)u_4=0$$

and

$$P : u_1-(mn-l)(lmn-1)u_4=0$$
$$Q : u_2-(nl-m)(lmn-1)u_4=0$$
$$R : u_3+(mn-l)(nl-m)u_4=0.$$

Suppose we are given a point, it is possible to find its polar plane with respect to a surface of the third order. If this surface degenerates into three planes (distinct), the problem is still possible. In like manner, if we are given the equations of three points, in four-point coördinates, it is always possible to determine uniquely the pole of a given plane with respect to this system of three points.

Let $v_1, v_2, v_3, v_4$ denote constant values of $u_1, u_2, u_3, u_4$, respectively; $\left.\frac{\partial F}{\partial u_1}\right]_{u_1=v_1} \equiv \frac{\partial F}{\partial v_1}$, and similarly in other cases.

The initial problem is to find a plane such that its pole with respect to the system of points, written in the symbolic form

$$LMN=0....(1)$$

is identical with its pole with respect to the second system of three points, written

$$PQR=0....(2)$$

The pole

$$\frac{\partial F}{\partial v_1}u_1+\frac{\partial F}{\partial v_2}u_2+\frac{\partial F}{\partial v_3}u_3+\frac{\partial F}{\partial v_4}u_4=0$$

of the plane $(v_1, v_2, v_3, v_4)$ with respect to the system (1) given by the equation

$$F(u_1, u_2, u_3, u_4)=0$$

has for its equation

$$[2lmnv_1v_4+n(l^2+m^2)v_2v_4+m(l^2n^2+1)v_3v_4+\lambda\mu^2\nu.v_4^2]u_1$$

$$+[2lmnv_2v_4+n(l^2+m^2)v_1v_4+l(m^2+n^2)v_3v_4+\lambda^2\mu\nu.v_4^2]u_2$$

$$+[2lmnv_3v_4+m(l^2n^2+1)v_1v_4+l(m^2+n^2)v_2v_4-\lambda\mu\nu^2.v_4^2]u_3$$

$$+[lmn(v_1^2+v_2^2+v_3^2)-3\lambda^2\mu^2\nu^2.v_4^2+n(l^2+m^2)v_1v_2+m(l^2m^2+1)v_1v_3$$
$$+l(m^2+n^2)v_2v_3+2\lambda\mu^2\nu.v_1v_4+2\lambda^2\mu\nu.v_1v_4+2\lambda^2\mu\nu.v_3v_4$$
$$-2\lambda\mu\nu^2.v_3v_4]u_4=0....(3)$$

where $\lambda, \mu, \nu \equiv mn-l, nl-m, lmn-1$, respectively. Also the pole

$$\frac{\partial \Phi}{\partial v_1}u_1 + \frac{\partial \Phi}{\partial v_2}u_2 + \frac{\partial \Phi}{\partial v_3}u_3 + \frac{\partial \Phi}{\partial v_4}u_4 = 0$$

of the plane $(v_1, v_2, v_3, v_4)$ with respect to the system (2) given by the equation

$$\Phi(u_1, u_2, u_3, u_4) = 0$$

has for its equation

$$\begin{aligned}
&[v_2v_3 - \mu\nu.v_3v_4 + \lambda\mu.v_2v_4 - \lambda\mu^2\nu.v_4^2]u_1 \\
&+[v_1v_3 - \lambda\nu.v_3v_4 + \lambda\mu.v_1v_4 - \lambda^2\mu\nu.v_4^2]u_2 \\
&+[v_1v_2 - \lambda\nu.v_2v_4 - \mu\nu.v_1v_4 + \lambda\mu\nu^2.v_4^2]u_3 \\
&+[-\lambda\nu.v_2v_3 - \mu\nu.v_1v_3 + 2\lambda\mu\nu^2.v_3v_4 + \lambda\mu.v_1v_2 - 2\lambda^2\mu\nu.v_2v_4 \\
&\quad -2\lambda\mu^2\nu.v_1v_4 + 3\lambda^2\mu^2\nu^2.v_4^2]u_4 = 0 \ldots . (4)
\end{aligned}$$

where $\lambda, \mu, \nu \equiv mn-l, nl-m, lmn-1$, respectively, as before.

Now it is evident by inspection that equations (3) and (4) are identical (aside from sign) of

$$v_1 = v_2 = v_3 = 0.$$

Accordingly the plane of the face $ABC$ of the fundamental tetrahedron $ABCD$ is such that its pole with respect to the point-triad (1) is coincident with its pole with respect to the point-triad (2).

Connect up the six points $L, M, N, P, Q, R$ by lines and planes in every possible way. Suppose the plane of $ABC$ to be intersected by the line $LM$ in the point $LM$, and by the plane $LMN$ in the line $LMN$; and so in the other cases. We obtain in this fashion a configuration in the plane of $ABC$, consisting of the fifteen ($\equiv {}_2C_6$) points $LM, LN, \ldots . QR$, and of the twenty ($\equiv {}_3C_6$) lines $LMN$, $LMP, \ldots . PQR$; and which is such that through each of the points there pass four of the lines, and on each of the lines lie three of the points. Thus the lines

$$\left.\begin{matrix} LMN \\ LMP \\ LMQ \\ LMR \end{matrix}\right\} \text{pass through the point } LM;$$

and the points

$$\left.\begin{matrix} LM \\ MN \\ NL \end{matrix}\right\} \text{lie on the line } LMN;$$

and so in other cases.

It will next be shown that six lines, denoted 1, 2, 3, 4, 5, 6, may be drawn in the plane of $ABC$, conditioned as follows:

$$(A)\begin{cases}\text{line (1) passes through the points } LP,\ MQ,\ NR\\ \text{“ (2) “ “ “ “ } LQ,\ MR,\ NP\\ \text{“ (3) “ “ “ “ } LR,\ MP,\ NQ\\ \text{“ (4) “ “ “ “ } LP,\ MR,\ NQ\\ \text{“ (5) “ “ “ “ } LQ,\ MP,\ NR\\ \text{“ (6) “ “ “ “ } LR,\ MQ,\ NP\end{cases}$$

For this purpose, represent any line in the plane of $ABC$ as the join of two points whose equations are

$$\left.\begin{aligned}\lambda_1 u_1+\mu_1 u_2+\nu_1 u_3=0\\ \lambda_2 u_1+\mu_2 u_2+\nu_2 u_3=0\end{aligned}\right\}$$

If this line meets the line $LP$, the join of the two points, whose equations are

$$\begin{aligned}L &: u_4=0\\ P &: u_1-(mn-l)(lmn-1)u_4=0\end{aligned}$$

we have the equation of condition

$$\begin{vmatrix}\lambda_1, & \mu_1, & \nu_1, & 0\\ \lambda_2, & \mu_2, & \nu_2, & 0\\ 0, & 0, & 0, & 1\\ 1, & 0, & 0, & -(mn-l)\,(lmn-1)\end{vmatrix}=0$$

or

$$\mu_1 : \mu_2=\nu_1 : \nu_2,$$

and hence the line in question may be written

$$\left.\begin{aligned}u_4=0\\ \mu_1 u_2+\nu_1 u_3=0\end{aligned}\right\}.$$

If this line meets the line $MQ$ also, we have the equation of condition

$$\begin{vmatrix}1, & 0, & 0, & 0\\ 0, & \mu_1, & \nu_1, & 0\\ m, & l, & lmn, & (mn-l)(nl-m)(lmn-1)\\ 0, & 1, & 0, & -(nl-m)(lmn-1)\end{vmatrix}=0$$

or

$$\mu_1 : \nu_1=1 : l$$

and hence the required line has for its equations

$$1:\left[\begin{aligned}u_1=0\\ u_2+lu_3=0\end{aligned}\right].$$

If now we write the equation of the point $N$ in the form

$$N : (nlu_1+mnu_2+lmnu_3)-(lmn-1)[u_3+(mn-l)(nl-m)u_4]=0$$

and note the equation of the point $R$

$$R : u_3+(mn-l)(nl-m)u_4=0,$$

it is evident that the equations of the line $NR$ may be written

$$NR: \left\{\begin{matrix} nlu_1+mnu_2+lmnu_3=0 \\ u_3+(mn-l)(nl-m)u_4=0 \end{matrix}\right\}$$

That the line 1 meets the line $NR$ is now obvious by inspection.

Determining in similar fashion the equations of the five remaining lines, we obtain

$$2: \left\{\begin{matrix} u_2=0 \\ mu_1+u_3=0 \end{matrix}\right\}$$

$$3: \left\{\begin{matrix} u_3=0 \\ nu_1+u_2=0 \end{matrix}\right\}$$

$$4: \left\{\begin{matrix} u_1=0 \\ lu_2+u_3=0 \end{matrix}\right\}$$

$$5: \left\{\begin{matrix} u_2=0 \\ u_1+mu_3=0 \end{matrix}\right\}$$

$$6: \left\{\begin{matrix} u_3=0 \\ u_1+nu_2=0 \end{matrix}\right\}$$

Now these six lines 1, 2, 3, 4, 5, 6 touch the conic given by the equation

$$lmn(u_1^2+u_2^2+u_3^2)+mn(l^2+1)u_2u_3+nl(m^2+1)u_3u_1+lm(n^2+1)u_1u_2=0.$$

This is most easily shown by putting $u_1$, $u_2$, $u_3$ in turn equal to zero; we obtain respectively,

$$mn(u_2+lu_3)(lu_2+u_3)=0,$$
$$nl(mu_1+u_3)(u_1+mu_3)=0,$$
$$lm(nu_1+u_2)(u_1+nu_3)=0.$$

Moreover it is clear from an inspection of the scheme (A) above that the points $LP$, $LQ$, $LR$; $MP$, $MQ$, $MR$; $NP$, $NQ$, $NR$ are the points 14, 25, 36; 35, 16, 24; 26, 34, 15, respectively, where 14, for example, denotes the meet of the lines 1 and 4; and so in other cases.

Conversely, starting from the six lines 1, 2, 3, 4, 5, 6 touching the above conic, and denoting the points 14, 25, 36; 35, 16, 24; 26, 34, 15 (which are indeed the vertices, and meets of opposite sides of the six-side 162435) in the man-

ner described above, then it is possible to complete the figure of the fifteen points $LM$, $LN$, ....$QR$ and of the twenty lines $LMN$, $LMP$, ....$PQR$, such that through each point pass four lines, and on each line lie three points, as detailed in the foregoing.

Of the fifteen points, nine, viz. the points $LP$, $LQ$, $LR$; $MP$, $MQ$, $MR$; $NP$, $NQ$, $NR$ are, as appeared above, points on two of the six lines 1, 2, 3, 4, 5, 6; the remaining points are $MN$, $NL$, $LM$; $QR$, $RP$, $PQ$. These are *Brianchon points*

| | | |
|---|---|---|
| $MN$ | of the six-side | 162435 |
| $NL$ | " | 152634 |
| $LM$ | " | 142536 |
| $QR$ | " | 152436 |
| $RP$ | " | 142635 |
| $PQ$ | " | 162534, |

for the point $MN$ is the meet of lines $MNP$, $MNQ$, $MNR \equiv MP$, $NP$; $MQ$, $NQ$; $MR$, $NR \equiv 35$, 26; 16, 34; 24, 15; that is, $MN$ is the Brianchon point of the six-side 162435; and similar reasoning verifies the above statements for the rest of the six-lines.

To summarize, we have two sets of three six-sides such that the Brianchon points of each set lie *in linea;* and the two lines so obtained together with the eighteen lines through the six Brianchon points, form a system of twenty lines passing by fours through fifteen points.

# DEPARTMENTS.

## SOLUTIONS OF PROBLEMS.

### ALGEBRA.

**154. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.**

Deduce the Sylvestrian Reciprocant of $ax^3 + 3bx^2y^2 + ay^3 + d = 0$.

**Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.**

Differentiating, dividing by 3, and combining, we have

$$a(x^2 + y^2\frac{dy}{dx}) + 2bxy(y + x\frac{dy}{dx}) = 0 \ldots . (1).$$

Repeating the operation, we have

$$a\left[x+2y\left(\frac{dy}{dx}\right)^2+y^2\frac{d^2y}{dx^2}\right]+2b\left[y^2+4xy\frac{dy}{dx}+x^2\left(\frac{dy}{dx}\right)^2+x^2y\frac{d^2y}{dx^2}\right]=0\ldots(2).$$

Eliminating $a$ and $b$ in equations (1) and (2), we have

$$\begin{vmatrix} x^2+y^2\frac{dy}{dx}, & xy\left(y+x\frac{dy}{dx}\right) \\ x+2y\left(\frac{dy}{dx}\right)^2+y^2\frac{d^2y}{dx^2}, & \left(y+2x\frac{dy}{dx}\right)^2+x^2y\frac{d^2y}{dx^2}-\left(\frac{dy}{dx}\right)^2 \end{vmatrix}=0,$$

which is the Sylvestrian Reciprocant of $ax^3+3bx^2y^2+ay^3+d=0$, since this function would have the same form if $x$ were the dependent and $y$ the independent variable.

167. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

A weight of $m$ pounds falls and is broken into $n$ pieces after which it is found that all weights, in pounds, from 1 to $m$ can be weighed. Find the weight of each piece. Apply when $m=121$, $n=5$.

I. Solution by the PROPOSER.

Let $x_1, x_2, x_3, \ldots x_n$ be the $n$ pieces. Then $x_1+x_2+x_3+\ldots+x_{n-1}+x_m=m$, $x_2-x_1=x_1+1$, or $x_2=2x_1+1$.

$x_3-x_2-x_1=x_2+x_1+1$, or $x_3=2x_2+2x_1+1=3(2x_1+1)=3x_2$.

$x_4-x_3-x_2-x_1=x_3+x_2+x_1+1$, or $x_4=2x_3+3x_2=9(2x_1+1)=9x_2$.

Generally, $x_r=3^{r-2}(2x_1+1)=3^{r-2}x_2$.

$\therefore x_1+x_2+x_3+\ldots+x_n=x_1+(2x_1+1)(1+3+9+27+\ldots+3^{n-2})=m$.

$\therefore x_1+(2x_1+1)(3^{n-1}-1)=2m$ or $x_1\frac{2m+1-3^{n-1}}{2.3^{n-1}}$.

$x_2=(2x_1+1)=\frac{2m+1}{3^{n-1}}$; $x_r=3^{r-2}x_2=\frac{2m+1}{3^{n-r+1}}$.

When $m=121$, and $n=5$, $x_1=1$, $x_2=3$, $x_3=9$, $x_4=27$, $x_5=81$.

II. Solution by FRANK L. GRIFFIN, Graduate Student, The University of Chicago.

Let $f(n)$=number of groupings of $n$ weights in two groups; then the maximum number giving one group a preponderence is $f(n)/2$.

Now, $f(n)=3^n-1\ldots$.(i) [for proof see below]. Hence, to weigh all weights, in pounds, from 1 to $m$ by using $n$ weights, it is necessary that

$$m\leq\frac{3^n-1}{2}\ldots.(ii).$$

By using the $n$ weights, 1, 3, 9, $\ldots 3^{n-1}$, all weights, in pounds, from 1 to

their sum, $S_n\left[=\frac{3^n-1}{2}\right]$ can be obtained; *i. e.* from 1 to [or beyond] $m$....(iii). [For proof of (iii) also see below].

Therefore, we arrive at the following solution:

A. When $m=\frac{3^n-1}{2}=S_n$.

The $n$ weights, 1, 3, 9, 27, ....$3^{n-1}$, satisfy both requirements: (a) sum of weights$=m$, (b) possibility of obtaining all weights from 1 to $m$.

B. When $m<\frac{3^n-1}{2}$.

The $(n-1)$ weights, 1, 3, 9, ....$3^{n-2}$, together with $(m-S_{n-1})$ as the other, satisfy requirements (a) and (b).

C. When $m>\frac{3^n-1}{2}$ there is no solution.

In the particular case where $m=121$, $n=5$, $m=\frac{3^n-1}{2}$, therefore the weights are 1, 3, 9, 27, and 81. [If $m$ had been between 41 and 121, $n$ would still have to be as much as 5, but the last weight would have been less than 81, (or$=m-40$)].

PROOF OF (i). Let $(A)....(B)$ be any grouping obtainable with $n$ weights; then by using one more weight, $w_{n+1}$, we get groupings as follows:

$$(A)....(B);\ (A+w_{n+1})....(B);\ (A)....(B+w_{n+1}).$$

Or for $(n+1)$ weights there are $3.f(n)$ groupings and in addition the two:

$$(w_{n+1})....(0) \text{ and } (0)....(w_{n+1}).$$

$\therefore f(n+1)=3.f(n)+2....(1)$.

Assume $f(n)=3^n-1$; then from (1), $f(n+1)=3^{n+1}-1$. Whence by induction, the formula holds for all values of $n$ as it is evidently true for $n=1$, $n=2$.

PROOF OF (iii). Assume that with $k$ weights, 1, 3, 9, ....$3^{k-1}$, all weights up to $S_k\left(=\frac{3^k-1}{2}\right)$ can be obtained. Then, since $w_{k+1}=3^k=2S_k+1$,

$$S_k+c=w_{k+1}-(s_k+1-c).$$

Therefore, any weight $S_k+c$ between $S_k$ and $S_{k+1}$ can be obtained by combining with $w_{k+1}$ a grouping $(S_k+1-c)$, which is not less than $S_k$ and hence is obtainable by weights, $w_1$, $w_2$, ....$w_k$. Therefore by using the first $(k+1)$ weights, all weights up to $S_{k+1}$ can be obtained.

Hence, by induction, since by weights $w_1$, $w_2$, all weights up to $s_2(=4)$

can be obtained, all weights from 1 to $\frac{3^n-1}{2}$ can be obtained by using the weights 1, 3, 9, 27, ....$3^{n-1}$.

Also solved by *J. SCHEFFER*, and *J. E. SANDERS*.
No solution of problem 168 has yet been received.

## GEOMETRY.

178. Proposed by JOHN M. ARNOLD, Crompton, R. I.

A cylinder thirty feet long and two feet in diameter is to be placed in a machinery car, the inside dimensions of which are eight feet wide and eight feet high. Find length of shortest car that will contain it.

Solution by the PROPOSER.

Consider the car standing on one end, which we shall call the base. Fig. 2.

If a projection of the cylinder be made on the base, each end will be projected into an ellipse with its minor axis on the diagonal $MN$ of the base.

Fig. 1 is a vertical plane taken on the line $MN$. [In Fig. 2, the points $H$, $D$, $E$, and $F$ correspond to the same points in Fig. 1.]

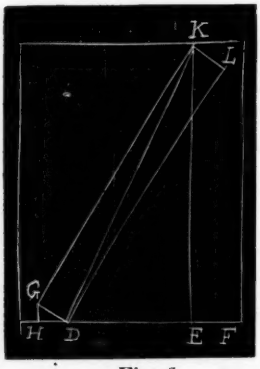


Fig. 1.

As the sides of the square in Fig. 2 are tangents to the equal ellipses, we have by Analytical Geometry $MO=\sqrt{(A^2+B^2)}$, where $A$ and $B$ are the semi-major and semi-minor axes,

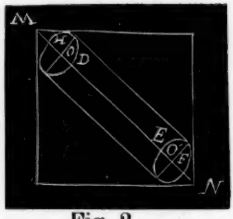


Fig. 2.

$$2\overline{MO}+\overline{OO'}=\overline{MN}=8\sqrt{2}....(1).$$

In the similar triangles $GDH$ and $DLF$, $GD : HD=DL : LF$, or $2A : 2B=30 : \frac{30B}{A}$. Then $DF=OO'=\sqrt{900-\frac{900B^2}{A^2}}$.

Substituting values of $OO'$ and $MO$ in (1),

$$2\sqrt{(A^2+B^2)}+\sqrt{900-\frac{900B^2}{A^2}}=8\sqrt{2}.$$

$A$=radius of cylinder=1. Substituting and reducing

$$12769B^4-21728B^2=-9184,$$

from which $B^2=.918904$ and $B=.9586$. Substituting the value of $B^2$ in the expression for $DF$, $DF=OO'=8.5432$. Then $DE=OO'-2B=6.626$, and $EK=\sqrt{[(30)^2+(2)^2-(6.626)^2]}=29.3274$ feet, or 29 feet 3.9288 inches, the length of car required.

Also solved by the late *P. H. PHILBRICK*, who obtained as a result, 29.168 feet.

## CALCULUS.

155. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College Defiance, Ohio.

Solve the differential equations:

$$(A).\quad \frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}=\sin 2x+\sin x-x. \qquad (B).\quad \frac{d^2y}{dx^2}+2\frac{dy}{dx}=\sin 2x+\sin x-x.$$

Solution by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, O., and LON C. WALKER, A. M., Leland Stanford University.

Using the symbolic method $(A)$ becomes $(D^4+2D^2)y=\sin 2x+\sin x-x$. The complementary function is $c_1+c_2x+c_3\cos\sqrt{2}x+c_4\sin\sqrt{2}x$, and the particular integral

$$=\frac{1}{D^2(D^2+2)}(\sin 2x+\sin x-x)=\frac{1}{D^2}\cdot\frac{1}{D^2+2}\sin 2x+\frac{1}{D^2}\cdot\frac{1}{D^2+2}\sin x-$$

$$\frac{1}{D^2+2}\cdot\frac{1}{D^2}x=\tfrac{1}{8}\sin 2x-\sin x-(2+D^2)^{-1}.\frac{x^2}{6}=\tfrac{1}{8}\sin 2x-\sin x-\frac{x^3}{12}+\tfrac{1}{4}x.$$

$\therefore\ y=c_1+c_2x+c_3\cos\sqrt{2}x+c_4\sin\sqrt{2}x+\tfrac{1}{8}\sin 2x-\sin x-\tfrac{1}{12}x^3$ ($\frac{1}{4}x$ being included in $c_2x$); and $(B)$ becomes $(D^2+2D)y=\sin 2x+\sin x-x$.

$\therefore$ The complementary function is $c_1+c_2e^{-2x}$, and the particular integral

$$=\frac{1}{D^2+2D}(\sin 2x+\sin x-x)=\frac{1}{D^2+2D}\sin 2x+\frac{1}{D^2+2D}\sin x-\frac{1}{D+2}\cdot\frac{1}{D}x$$

$$=\frac{1}{2D-4}\sin 2x+\frac{1}{2D-1}\sin x-(2+D)^{-1}.\tfrac{1}{2}x^2$$

$$=\tfrac{1}{2}\cdot\frac{D+2}{D^2-4}\sin 2x+\frac{2D+1}{4D^2-1}\sin x-(\tfrac{1}{2}-\tfrac{1}{4}D+\tfrac{1}{8}D^2)\tfrac{1}{2}x^2$$

$$=-\tfrac{1}{16}(D+2)\sin 2x-\tfrac{1}{5}(2D+1)\sin x-\tfrac{1}{4}x^2+\tfrac{1}{4}x-\tfrac{1}{8}$$

$$=-\tfrac{1}{8}\cos 2x-\tfrac{1}{8}\sin 2x-\tfrac{2}{5}\cos x-\tfrac{1}{5}\sin x-\tfrac{1}{4}x^2+\tfrac{1}{4}x-\tfrac{1}{8}.$$

$\therefore\ y=c_1+c_2e^{-2x}-\frac{1}{8}(\cos 2x+\sin 2x)-\frac{2}{5}\cos x-\frac{1}{5}\sin x-\frac{1}{4}x^2+\frac{1}{4}x$, ($-\frac{1}{8}$ being included in the term $c_1$).

Also solved by *J. SCHEFFER*, *W. W. LANDIS*, *G. W. GREENWOOD*, and *WILLIAM HOOVER*.

156. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the volume common to the two solids $x^2+y^2+z^2=a^2$ and $xz^2=(a-x)(x^2+y^2)$.

I. Solution by the PROPOSER.

The limits of $z$ are $z=\sqrt{\frac{a-x}{x}(x^2+y^2)}$ to $z=\sqrt{(a^2-x^2-y^2)}$.

Eliminating $z$, $y=\sqrt{[x(a-x)]}$.

$\therefore$ The limits of $y$ are 0 and $\sqrt{[x(a-x)]}=y'$; the limits of $x$ are 0 and $a$.

$$\therefore V=4\int_0^a\int_0^{y'}\left[\sqrt{(a^2-x^2-y^2)}-\sqrt{\frac{a-x}{x}(x^2+y^2)}\right]dxdy,$$

$$=2\int_0^a\left[(a^2-x^2)\sin^{-1}\sqrt{\frac{x}{a+x}}-x\sqrt{(ax-x^2)}\log\left(\frac{\sqrt{a}+\sqrt{(a-x)}}{\sqrt{x}}\right)\right]dx.$$

Let $x=a\tan^2\theta$ in the first term, and $x=a\sin^2\theta$ in the second term.

$$\therefore V=4a^3\int_0^{\frac{1}{4}\pi}\theta(1-\tan^4\theta)\tan\theta\sec^2\theta d\theta-4a^3\int_0^{\frac{1}{4}\pi}\sin^4\theta\cos^2\theta\log\left(\frac{1+\cos\theta}{\sin\theta}\right)d\theta$$

$$=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3+\tfrac{1}{4}a^3\int_0^{\frac{1}{4}\pi}\log(\tan\tfrac{1}{2}\theta)d\theta.$$

$$\int_0^{\frac{1}{4}\pi}\log(\tan\tfrac{1}{2}\theta)d\theta=\tfrac{1}{2}\int_0^{\frac{1}{4}\pi}\log\left(\frac{1-\cos\theta}{1+\cos\theta}\right)d\theta=-\int_0^{\frac{1}{4}\pi}(\cos\theta+\tfrac{1}{3}\cos^3\theta+\tfrac{1}{5}\cos^5\theta+.....)d\theta$$

$$=-2\int_0^{\frac{1}{4}\pi}(\cos\theta+\tfrac{1}{3}\cos3\theta+\tfrac{1}{5}\cos5\theta+....)d\theta=-2(1-1/3^2+1/5^2-1/7^2+1/9^2-...)$$

$$=-16\left[\frac{1}{1^2.3^2}+\frac{3}{5^2.7^2}+...+\frac{2n-1}{(4n-3)^2(4n-1)^2}\right]$$

$$=-1.832 \text{ nearly}=-16[.114488335].$$

$\therefore V=\frac{2}{3}\pi a^3-\frac{5}{4}a^3-.458a^3=\frac{2}{3}\pi a^3-1.708a^3$.

II. Solution by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Transforming the two surfaces to polar coördinates, the equations of the first are $\rho=a$, and of the second, $\rho=a\sin\phi\sec\theta$. The equations of the intersection of the two surfaces are $\left.\begin{matrix}\rho=a\\ \sin\phi=\cos\theta\end{matrix}\right\}$.

The volume common to the two solids bounded by the surfaces is

$$V=4\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}\int_{a\sin\phi\sec\theta}^{\frac{1}{2}\pi}\rho^2\sin\phi d\phi d\theta d\rho=\tfrac{4}{3}a^3\int_0^{\frac{1}{2}\pi}\int_0^{\frac{1}{2}\pi-\phi}(1-\sin^3\phi\sec^3\theta)\sin\phi d\phi d\theta,$$

$$=\tfrac{4}{3}a^3[\tfrac{1}{2}\pi-\tfrac{1}{16}\tfrac{5}{}+\tfrac{3}{16}\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3+\tfrac{1}{4}a^3\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi],$$

$$=\tfrac{2}{3}\pi a^3-\tfrac{5}{4}a^3-\tfrac{1}{4}a^3\sum_{n=1}^{n=\infty}\frac{2n-1}{[4n-3]^2\ [4n-1]^2}=.386a^3,\text{ nearly.}$$

On the integration of $\int_0^{\frac{1}{2}\pi}\log\tan\frac{\phi}{2}d\phi$, see the remarks on Prize Problem, No. 123, Calculus.

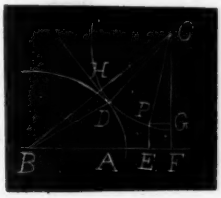
157. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Two equal ellipses are tangent to each other at the vertices of the major axes. If one of them be rolled on the other; find (1) the equation and area of the curve described by the vertex, and (2) by the center.

Solution by G. B. M. ZERR, A.M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa., and the PROPOSER.

Let $a$ and $b$ be the semi-axes of both ellipses; $B$ the center of the fixed ellipse: $C$, the center of the rolling ellipse; $P$, its vertex; and $D$, the point of contact; $e^2=(a^2-b^2)/a^2$. Let $BE=x$, $BF=m$, $PE=y$, $CF=n$, and $\angle ABC=\angle PCB=\theta$.

Then, $BC=2a\sqrt{(1-e^2\sin^2\theta)}$=twice the length of the perpendicular from $B$ on the tangent at $D$.




$m=2a\cos\theta\sqrt{(1-e^2\sin^2\theta)}$, $n=2a\sin\theta\sqrt{(1-e^2\sin^2\theta)}$,

$x=m-PG=2a\cos\theta\sqrt{(1-e^2\sin^2\theta)}-a\cos2\theta$,

$y=n-CG=2a\sin\theta\sqrt{(1-e^2\sin^2\theta)}-a\sin2\theta$.

$x^2+y^2=r^2=a^2+4a^2(1-e^2\sin^2\theta)-4a^2\cos\theta\sqrt{(1-e^2\sin^2\theta)}$,

the equation of the locus of the vertex.

$$\text{Area}=a^2\int_0^{\pi}[5-4e^2\sin^2\theta-4\cos\theta\sqrt{(1-e^2\sin^2\theta)}]d\theta=\pi a^2(5-2e^2)=\pi(3a^2+2b^2).$$

$BC^2=\rho^2=4a^2(1-e^2\sin^2\theta)$, the equation of the locus of the center.

$$\text{Area}=4a^2\int_0^{\pi}(1-e^2\sin^2\theta)d\theta=2\pi a^2(2-e^2)=2\pi(a^2+b^2).$$

Also solved by J. SCHEFFER, and G. W. GREENWOOD.

## MECHANICS.

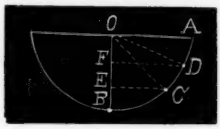
147. Proposed by W. J. GREENSTREET, M. A., Editor of The Mathematical Gazette, Stroud, England.

A particle mass $m$ is attached to one end of a string, the other end of which is fixed. It is projected horizontally with such a velocity that it would rise to a position in which

the string would be horizontal. But on its upward path it meets an inelastic particle mass $m'$ and the height to which it rises is diminished by $1/p$th of what it would have risen. Find $m'$, and the tensions of the string just after collision and at the greatest height of the particle.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $B$ be the point of projection of particle mass $m$; $C$, the position of $m'$ supposed at rest; $D$, the point at which $m$ stops after impact with $m'$. Let $\angle COB=\theta$, $BO=a$.. Then $OF=a/p$, $FB=a(p-1)/p$, $OE=a\cos\theta$. Velocity of projection$=\sqrt{(2ga)}$, velocity of $m$ at $C$ at moment of impact$=\sqrt{(2ga\cos\theta)}$.




$\therefore\ m\sqrt{(2ga\cos\theta)}=v'(m+m')$.

But $v'=\sqrt{(2g.FE)}=\sqrt{[(2ga/p)(p\cos\theta-1)}$.

$\therefore\ m\sqrt{(2ga\cos\theta)}=(m+m')\sqrt{[(2ga/p)(p\cos\theta-1)}$.

$$\therefore\ m'=\frac{m\{\sqrt{[\cos\theta]}-\sqrt{[(p\cos\theta-1)/p]}\}}{\sqrt{[(p\cos\theta-1)/p]}}.$$

Let $(m+m')g=W$. The tension is composed of the tension due to acceleration imparted by the central force plus the tension due to gravity. Let $T=$ tension just after impact, $t=$tension at highest point.

Then $T=\dfrac{Wv'^2}{ga}+W\cos\theta=(2W/p)(p\cos\theta-1)+W\cos\theta=3W\cos\theta-2W/p$.

Since velocity is zero at $D$, $t=W\cos DOB=W/p$. $\therefore\ t=W/p$.

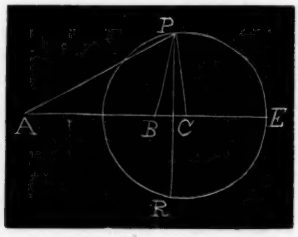
As $m'$ is inelastic, it is supposed that $m'$ coalesces with $m$.

148. Proposed by G. H. HARVILL, A. M., Malakoff, Texas.

Show that a law of density for points in space may be assumed such that the joint mass of any two points which are *electrical images* of each other in respect to a given sphere may be constant, and that their centers of gravity should lie on the surface of the sphere.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $A$, $B$ be the electrical images, $a=$radius of sphere, $C$ its center, $AC=x$, $BC=y$. Let $e=$charge of electricity situated at $A$; and let the sphere be insulated and have a charge $-(ea^3/x^3)$. Let $PR$ be the line of no electrification. Let $m=$mass of each point on surface of sphere; $u=$mass of point at $A$; $v=$mass of point at $B$; $\angle BAP=\theta$. Now $AP=AC$, $BP=PC$ (since $PR$ is line of no electrification). Also




$$u+v=2m\ldots.(1),$$
$$u(x-a)=v(a-y)\ldots.(2).$$

From triangles $ACP$ and $ABP$ we have, since $AP=AC=x$, $BP=PC=a$,

$$\cos\theta=\frac{2x^2-a^2}{2x^2}=\frac{x^2+(x-y)^2-a^2}{2x(x-y)}.$$

$\therefore x=a^2/y$. Let $y=a/n$. $\therefore x=an=n^2y$. From (1) and (2),

$$u=\frac{2m}{1+n},\quad v=\frac{2mn}{1+n},\quad u=\frac{v}{n}.$$

Hence, the law: The distance, from the center of the sphere of the point without varies as the square of the distance of the point within from the center of the sphere. The mass of the point without varies directly as the mass of the point within and inversely as the distance from the center.

## DIOPHANTINE ANALYSIS.

103. Proposed by HARRY S. VANDIVER, Bala, Pa.

Find some solutions of $x^3+ay^3=z^3$ (for $x$, $y$, and $z$) and show that there is an infinite number of solutions corresponding to each integral value of $a$.

Solution by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

For generality we may write

$$x^3+mx^2y+nxy^2+ay^3=x^3....(1).$$

Let us consider the following products.

$$(u+av+a^2w)(u+\beta v+\beta^2w)(u+\gamma v+\gamma^2w)....(2), \text{ and}$$
$$(u'+av'+a^2w')(u'+\beta v'+\beta^2w')(u'+\gamma v'+\gamma^2w')....(3),$$

where $\alpha$, $\beta$, $\gamma$ are the roots of $\theta^3-m\theta^2+n\theta-a=0....(4)$,

Then $\Sigma\alpha=m$, $\Sigma\alpha\beta=n$, and $\alpha\beta\gamma=a$. Since $\alpha$ is a root of (4), we have

$$\alpha^3=m\alpha^2-n\alpha+a, \text{ or}$$
$$\alpha^4=m\alpha^3-n\alpha^2+a\alpha=(m^2-n)\alpha^2-(mn-a)\alpha+am....(5).$$

Hence the product of the first factor in (2) by the first factor in (3), is $U+\alpha V+\alpha^2W$, where

$$U=uu'+a(vw'+v'w)+amww',$$
$$V=uw'+u'v-n(vw'+v'w)-(mn-a)ww',$$
$$W=uw'+u'w+vv'+m(vw'+v'w)+(m^2-n)ww',$$

and where the values of $\alpha^3$ and $\alpha^4$ were substituted from (5).

In like manner, we find that the product of the six factors in (2) and (3), to be

$$(U+\alpha V+\alpha^2W)(U+\beta V+\beta^2W)(U+\gamma V+\gamma^2W)....(6).$$

Now put $u'=u, v'=v, w'=w$; then (6) becomes $(u+av+a^2w)^2(u+\beta v+\beta^2 w)^2 \times(u+\gamma v+\gamma^2 w)^2$, where

$$U=u^2+2avw+amw^2....(A),$$
$$V=2uv-2nvw-(mn-a)w^2....(B),$$
$$W=2uw+v^2+2mvw+(m^2-n)w^2....(C).$$

Whence we have the general solution of

$$U^3+mU^2V+(m^2-2n)U^2W+nUV^2+(mn-3a)UVW+(n^2-2am)UW^2+aV^3 +amV^2W+anVW^2+a^2W^3=z^2....(7).$$

To make (7) applicable to (1), we set $U=x$, $V=y$, $W=0$. Then from $(A)$, $(B)$, and $(C)$, we find

$$u=-\frac{v^2+2mvw+(m^2-n)w^2}{2w},$$
$$x=u^2+2avw+amw^2,$$
$$y=2uv-2nvw-(mn-a)w^2,$$

in which $v$ and $w$ may be chosen at pleasure. When $m=n=0$,

$$u=-\frac{v^2}{2w}, \quad x=u^2+2avw, \quad y=2uv+aw^2,$$

Hence, since $u$ and $v$ may be chosen at pleasure, there is an infinite number of solutions corresponding to each integral value of $a$.

Excellent solutions were also received from *G. B. M. ZERR*, and the *PROPOSER*.

## AVERAGE AND PROBABILITY.

126. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find the average ellipse inscribed in a triangle, so that the sides of the triangle are tangent to the ellipse.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $ABC$ be the triangle; $DEF$ the inscribed ellipse; $D$, $E$, $F$ the points where the sides touch the ellipse; $G$, the point of concurrence of $AD$, $BE$, $CF$; $GH=l$, $GK=m$, $GL=n$; $l$, $m$, $n$, the perpendiculars from $G$ on the sides; $BA$, $BC$, the axes of coördinates; $a, b, c$ the sides of the triangle; $\triangle$, its area; $P$, the area of the ellipse; and $Q$, the average area required. Then

$$[(blm+cln)x+(aln+bmn)y-acln]^2-4b^2lm^2nxy=0,$$

is the equation to the ellipse.

$$\therefore P=\frac{\pi a^2bc^2lmn\sin B}{2\sqrt{[(ablm+acln+bcmn)^3]}}=\frac{\pi abc\triangle lmn}{\sqrt{[(ablm+acln+bcmn)^3]}}.$$

But $al+bm+cn=2\triangle$.

$$\therefore P=\frac{\pi\triangle acln[2\triangle-al-cn]}{\sqrt{[(2\triangle al+2\triangle cn-acln-a^2l^2-c^2n^2)^3]}}.$$

From the equation to the ellipse, we get

$$BD=u=\frac{acn}{bm+cn}=\frac{acn}{2\triangle-al},\quad BF=v=\frac{acl}{al+bm}=\frac{acl}{2\triangle-cn}.$$

$$\therefore al=\frac{2\triangle v(a-u)}{ac-uv},\quad cn=\frac{2\triangle u(c-v)}{ac-uv}.$$

$$\therefore P=\frac{\pi\triangle uv\sqrt{[(a-u)(c-v)]}}{\sqrt{[(av+cu-uv)^3]}}.\quad \text{Let } a-u=t,\ c-v=z.$$

$$\therefore P=\frac{\pi\triangle[a-t][c-z]\sqrt{[tz]}}{\sqrt{[(ac-tz)^3]}}.$$

$$\therefore Q=\int_0^a\int_0^c Pdtdz/\int_0^a\int_0^c dtdz=\frac{\pi\triangle}{ac}\int_0^a\int_0^c\frac{(a-t)(c-z)\sqrt{(tz)}dtdz}{\sqrt{[(ac-tz)^3]}}.$$

Let $tz=a\sin^2\theta$, $\theta'=\sin^{-1}\sqrt{(t/a)}$.

$$\therefore Q=\frac{2\pi\triangle}{a}\int_0^a\int_0^{\theta'}\frac{(a-t)(t-a\sin^2\theta)\sin^2\theta dtd\theta}{t^2\cos^2\theta}$$

$$=\frac{\pi\triangle}{a}\int_0^a\left(\frac{a-t}{t^2}\right)\{3a\sin^{-1}\sqrt{(t/a)}-2t\sin^{-1}\sqrt{(t/a)}-3\sqrt{[t(a-t)]}\}dt.$$

Let $t=a\sin^2\varphi$.

$$\therefore Q=2\pi\triangle\int_0^{\frac{1}{2}\pi}(3\varphi-2\varphi\sin^2\varphi-3\sin\varphi\cos\varphi)\cot^3\varphi d\varphi=\tfrac{1}{2}\pi^2\triangle(7-10\log^2)$$

**127.** Proposed by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

What is the probable error of the volume of a rectangular parallelopiped whose edges measured by the repeated application of a unit of measure are found to be $a$, $b$, $c$, supposing that the probable error of a line so measured whose length is found to be $l$ is $r\sqrt{l}$?

Solution by the PROPOSER.

The probable error for $a$, $=r\sqrt{a}$; for $b$, $r\sqrt{b}$; for $c$, $r\sqrt{c}$.

$\therefore$ Error of volume in length$=bcr\sqrt{a}$.

Error of volume in width$=acr\sqrt{b}$.

Error of volume in thickness$=abr\sqrt{c}$.

The probable error of volume=square root of the sum of the squares of these three errors.

$\therefore$ Probable error$=\sqrt{[(ab^2c^2+a^2bc^2+a^2b^2c)r^2]}=r\sqrt{[abc(ab+ac+bc)]}$.

## MISCELLANEOUS.

124. Proposed by J. W. YOUNG, Graduate Student, Cornell University, Ithaca, N. Y.

Prove that the general value of $\theta$, which satisfies the equation

$$(\cos\theta+i\sin\theta)(\cos2\theta+i\sin2\theta)\ldots\text{to } n \text{ factors}=1 \text{ is } \frac{4m\pi}{n(n+1)};$$

where $m$ is any integer ($i=\sqrt{-1}$).

Solution by G. W. GREENWOOD, A. M., McKendree College, Lebanon, Ill.; LON C. WALKER, A. M., Leland Stanford Jr. University, Cal., and J. SCHEFFER, A. M., Hagerstown, Md.

$$1=(\cos\theta+i\sin\theta)(\cos2\theta+i\sin2\theta)\ldots(\cos n\theta+i\sin n\theta)=(\cos\theta+i\sin\theta)^{1+2+\ldots+n}$$

$$=(\cos\theta+i\sin\theta)^{\frac{1}{2}[n(n+1)]}=\cos\frac{n(n+1)\theta}{2}+i\sin\frac{n(n+1)\theta}{2}.$$

$$\therefore \frac{n(n+1)\theta}{2}=2m\pi, \text{ where } m \text{ is any integer; } i.\ e.\ \theta=\frac{4m\pi}{n(n+1)}.$$

Also solved by G. B. M. ZERR.

125. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Assume $m=nt+\varepsilon-\omega$, thus giving $v=m+e\sin v$ as the relation connecting the mean and eccentric anomalies, then express $x=a\cos v$, $y=b\sin v$, and $r=a(1-e\cos v)$ by a Fourier series in terms of $m$.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

If $y_1=z+x\varphi((y)$, we get by Lagrange's Theorem,

$$f(y_1)=f(z)+x\varphi(z)f'(z)+\frac{x^2}{1.2}\frac{d}{dz}\{[\varphi(z)]^2f'(z)\}+\frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2\{[\varphi(z)]^3f'(z)\}+$$

etc., etc.

From $v=m+e\sin v$, $y_1=v$, $z=m$, $x=e$, $\varphi(y)=\sin v$.

Now $f(v)=v$ and $f'(v)=1$.

$$\therefore\ y_1 = z + x\sin z + \frac{x^2}{1.2}\frac{d}{dz}(\sin^2 z) + \frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2(\sin^3 z) + \text{ etc.}$$

$$\therefore\ y_1 = z + x\sin z + \frac{x^2}{1.2}\frac{d}{dz}\left(\frac{1-\cos 2z}{2}\right) + \frac{x^3}{1.2.3}\left(\frac{d}{dz}\right)^2\left(\frac{3\sin z - \sin 3z}{4}\right)$$

$$+ \frac{x^4}{1.2.3.4}\left(\frac{d}{dz}\right)^3\left(\frac{3 - 4\cos 2z + \cos 4z}{8}\right) + \text{ etc.}$$

$$= z + x\sin z + \tfrac{1}{2}x^2\sin 2z + \tfrac{1}{8}x^3(3\sin 3z - \sin z).$$

$$\therefore\ v = m + e\sin m + \tfrac{1}{2}e^2\sin 2m + \tfrac{1}{8}e^3(3\sin 3m - \sin m)$$

$$+ \tfrac{1}{6}e^4(2\sin 4m - \sin 2m) + \text{ etc.} = m + e\sin v.$$

$$\therefore\ \sin v = \sin m + \tfrac{1}{2}e\sin 2m + \tfrac{1}{8}e^2(3\sin 3m - \sin m) + \tfrac{1}{6}e^3(2\sin 4m - \sin 2m) + \text{etc.}$$

To develop $(1 - e\cos v)$ in terms of $m$: Let $f(y_1) = 1 - e\cos y_1$, $f'(y_1) = e\sin y_1$.

$$\therefore\ 1 - e\cos y_1 = (1 - e\cos z) + x\sin z(e\sin z) + \frac{x^2}{1.2}\frac{d}{dz}(\sin^2 z.e\sin z) + \text{ etc.}$$

Performing the operations as before we get after substituting $v$ for $y_1$, $m$ for $z$, $e$ for $x$,

$$1 - e\cos v = 1 - e\cos m + \tfrac{1}{2}e^2(1 - \cos 2m) + \tfrac{1}{8}e^3(3\cos m - 3\cos 3m)$$

$$+ \tfrac{1}{4}e^4(\cos 4m - \cos 2m) + \text{ etc.}$$

$$\therefore\ \cos v = \cos m + \tfrac{1}{2}e(\cos 2m - 1) + (3e^2/8)(\cos 3m - \cos m)$$

$$+ \tfrac{1}{4}e^3(\cos 2m - \cos 4m) + \text{ etc.}$$

$$\therefore\ \sin v = A\sin m + B\sin 2m + C\sin 3m + D\sin 4m + \ldots.$$
$$\cos v = -\tfrac{1}{2}e + A_1\cos m + B_1\cos 2m + C_1\cos 3m + D_1\cos 4m + \ldots.$$

where $A$, $B$, $C$, $D$, $\ldots.A_1$, $B_1$, $C_1$, $D_1$, $\ldots.$ are each a series in powers of $e$.

$$\therefore\ x = a\cos v = -\tfrac{1}{2}ae + aA_1\cos m + aB_1\cos 2m + aC_1\cos 3m + aD_1\cos 4m + \ldots.$$
$$y = b\sin v = bA\sin m + bB\sin 2m + bC\sin 3m + bD\sin 4m + \ldots.$$
$$r = a(1 - e\cos v) = a[1 + \tfrac{1}{2}e^2 - eA_1\cos m - eB_1\cos 2m - eC_1\cos 3m - eD_1\cos 4m + \ldots.].$$

**126.** Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The declination of a certain fixed star if 12° 40′. Its altitude was observed one day to be 16° 40′. Three hours and twenty-four minutes later it was found to be 40° 20′. Find the latitude of the place of observation.

Solution by G. B. M. ZERR, A. M., Ph.D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Let $\delta=12^\circ\ 40'$, $\varphi$=latitude of observation, $a=16^\circ\ 40'$, $a'=40^\circ\ 20'$, $\mu=51^\circ$ =3 hours, 24 minutes, $h$=hour angle.

$$\text{Then } \cos h=\frac{\sin a-\sin\varphi\sin\delta}{\cos\varphi\cos\delta}=x,\quad \cos(h-\mu)=\frac{\sin a'-\sin\varphi\sin\delta}{\cos\varphi\cos\delta}=y.$$

Eliminating $h$, $\sin^2\mu=x^2-2xy\cos\mu+y^2$. Substituting values of $x$ and $y$,

$$\cos^2\varphi\cos^2\delta\sin^2\mu=2\sin^2\varphi\sin^2 d(1-\cos\mu)+\sin^2 a+\sin^2 a'-2\sin a\sin a'\cos\mu$$
$$-2\sin\varphi\sin\delta(1-\cos\mu)(\sin a+\sin a').$$

$$\therefore \sin^2\varphi[\sin^2\mu+\sin^2\delta(1-\cos\mu)^2]-2\sin\varphi\sin\delta(1-\cos\mu)(\sin a+\sin a')$$
$$=\cos^2\delta\sin^2\mu-\sin^2 a-\sin^2 a'+2\sin a\sin a'\cos\mu.$$

Let $\sin^2\mu+\sin^2\delta(1-\cos\mu)^2=A=.610569.$
$\sin\delta(1-\cos\mu)(\sin a+\sin a')=B=.075921.$
$\cos^2\delta\sin^2\mu-\sin^2 a-\sin^2 a'+2\sin a\sin a'\cos\mu=C=.307394.$

$$\therefore A\sin^2\varphi-2B\sin\varphi=C.$$
$$\therefore \sin\varphi=\frac{B\pm\sqrt{(AC+B^2)}}{A}=.844702, \text{ or } -.596013.\quad \varphi=57^\circ\ 38'\ 37''.$$

## PROBLEMS FOR SOLUTION.

### ARITHMETIC.

166. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

If I sell one of my farms for $\$A$,=\$4500, and the other for $\$B$,=\$1800, I will gain $p\%$, =5%, on cost of both; but if I sell the dearer farm for $\$C$,=\$4000, and the other at cost, I will lose $p\%$,=5%. Find the cost of each farm.

### AVERAGE AND PROBABILITY.

138. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Chemistry and Physics, The Temple College, Philadelphia, Pa.

Find the average area of (1) triangle, (2) quadrilateral, (3) pentagon, (4) hexagon, formed by taking (1) three, (2) four, (3) five, (4) six random points on the circumference of a given circle radius $a$.

139. Proposed by L. C. WALKER, A. M., Graduate Student, Leland Stanford Jr. University, Cal.

Four points are taken at random on the surface of a given sphere; find the average volume of the tetrahedron formed by the planes passing through the points taken three and three.

## MISCELLANEOUS.

131. Proposed by SAUL EPSTEEN, Ph. D., Professor of Mathematics, University of North Carolina.

Find a power series for $\pi^{nx}$ ($n$=any integer).

132. Proposed by M. A. GRUBER, A. M., War Department, Washington, D. C.

Six officers of different grades (1, 2, 3, 4, 5, 6) from each of six branches of the army ($a, b, c, d, e, f$) are to be arranged in a square so that each rank and each file shall have an officer of each grade and each branch. Can it be done? If not, prove it. The arrangement of five officers of each kind is easy.

133. Proposed by HARRY S. VANDIVER, Bala, Pa.

If a group $G$ of order $mn$ has a subgroup $H$ of order $n$, and if $n$ has no prime factor which is less than $m$, show that $H$ must be a self-conjugate sub-group. (Frobenius.)

134. Proposed by F. P. MATZ, Sc. D., Ph. D., Professor of Mathematics and Astronomy in Defiance College, Defiance, Ohio.

Give a complete solution of the Jacobian equation $\kappa^2 \mathrm{sn}^4 u + 2\kappa^2 \mathrm{sn}^2 u + 1 = 0$.

## *BOOKS AND PERIODICALS.*

*Accounting and Business Practice.* By John H. Moore, Commercial Department, Boston High School, and George W. Miner, Commercial Department, Westfield (Mass.) High School. 8vo. Cloth, 400 pages. List price, $1.50; mailing price, $1.55. Boston: Ginn & Co.

"Accounting and Business Practice is a thorough, practical, and comprehensive text for the use of teachers and students of book-keeping. It is intended for use in high schools, private schools, and all institutions where accounting is taught, and is well adapted for teaching by correspondence. Attractive blank books and business forms accompany the text. The work is arranged in the following general divisions: Introductory, presenting a series of definite lessons for beginners embracing lesson outlines, exercises for class drills, two brief sets in elementary accounting, and two sets for business practice. Intermediate, presenting the subject of drafts, three sets of more advanced business practice, introducing the use of special columns, and auxiliary ledgers. Advanced, containing three sets, single entry, corporation accounting (a set on manufacturing), and banking.

A few special features: 1. The work is complete in itself and is not accompanied by a system of vouchers. 2. The work is elastic and may be used in the study of theory only, or of theory and business practice. 3. Financial statements are given in connection with all the different sets. 4. Class exercises are given in connection with every important subject introduced. 5. The text is accompanied by a Teachers' Manual giving a large amount of material for class drills in practical accounting, arranged in a series of lessons carefully graded."

*The Universal Solution* for Numerical and Literal Equations by which the Roots of Equations of All Degrees can be expressed in terms of their Coefficients. By M. A. McGinnis. 8vo. Cloth, 194 pages. Price, $2.00. Kansas City: The Mathematical Book Co.

We cannot praise this book very highly for the merit it possesses, since the really meritorious part of the book deals with matter quite irrelevant to what the work professes

to discuss, and to solve, viz., the solution of the general equation of the fifth and other degrees. The book contains some ingeneous methods of solving certain numerical equations, but because of these methods it should have received a more modest title.

Mr. McGinnis's solution of the Sixth Degree is quite erroneous. Mr. W. M. H. Woodward, pp. 153-150, professes to have demolished the proof of the impossibility of solving the general quantic by radicals given in Serret's *Algebra Supérieure*. But judging from his conclusion, it appears that Mr. Woodward does not understand the argument put forth in *Algebra Supérieure*. B. F. F.

*An Elementary Text-book on the Differential and Integral Calculus.* By William H. Echols, Professor of Mathematics in the University of Virginia. 8vo. Cloth, x+480 pages. Price, $2.00. New York: Henry Holt & Co.

In this work are very ably treated many interesting subjects not to be found in any other American text-book on the Calculus.

In order to form a connecting link between Algebra and the Calculus, an Introduction presents in an admirable way the fundamental and essential features of Arithmetic and Algebra. In the Introduction are defined and explained such ideas as *absolute number, the absolute-number continuum, the real-number system, the limit of a variable*, etc., ideas upon which rest the whole structure of the Calculus. Throughout the work, in establishing the principles much attention is given to the applications of those principles. An unusually large number of interesting and well selected problems are appended to each section.

The work is divided into two books. Book I treats of functions of one variable, and is divided into four parts. Part I embraces the Principles of the Differential Calculus; Part II applies these principles to Geometry; Part III establishes the Principles of the Integral Calculus, and Part IV embraces the application of these principles. Book II treats of functions of more than one variable. It is divided into three parts. The first part which is Part V of the entire work, embraces Principles and Theory of Differentiation; Part VI applies the principles to surfaces, and Part VII treats of Integration of more than one Variable and Multiple Integration.

Part VI extends the principles of the Calculus to surfaces. Here we have such problems as: To find the principal radii to a surface; To determine the umbilics on a surface, etc. Also here is discussed pretty fully such subjects as spherical curvature, envelopes of surfaces, etc.

The author, while acknowledging that the introduction of a new symbolism is always objectionable, yet feels called upon to introduce the "English pound" mark for the symbol of passing to the limit. This is certainly desirable. But personally we prefer Professor Oliver's symbol, $\doteq$, for "converging to" or approaches, to Professor Echol's symbol, (=), which he introduces to mean the same thing.

The work is a most valuable addition to the many meritorious books on the same subject which have appeared in recent years. B. F. F.

*The School Visitor.* Published by John S. Royer & Sons, No. 247 North 17th Street, Columbus, Ohio. Price per year, $1.00, payable in advance.

The Mathematical Department is full of good problems for the teacher of Arithmetic, Algebra and Geometry. Mr. Royer is the author of a Higher Mental Arithmetic, a Geography, and several other books of great interest and value to teachers. B. F. F.

## Errata.

Vol. IX, page 207, problem 106, Diophantine Analysis, for "rational triangle" read, *rational right triangle.*

Vol. IX, page 264, last line of solution of problem 100, for "determinate" read, *indeterminate*; page 265, equation for "$Ay'q+$" read $Ay'+q$.

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# THE GROUP GENERATED BY CENTRAL SYMMETRIES, WITH APPLICATION TO POLYGONS.

By DR. EDWARD KASNER.

The object of this article is to generalize the following well known theorem of elementary geometry: If the mid-points of the consecutive sides of any quadrilateral are joined the resulting figure is a parallelogram. In the case of a triangle the corresponding construction gives a triangle not having any peculiar property. The question therefore arises as to whether or not in case of polygons of more than four sides there is any theorem analogous to that concerning the quadrilateral. It will be shown that, in this respect, there is an essential distinction between polygons of an even number of sides, and those of an odd number of sides. This depends fundamentally upon the character of the group which is discussed in §1.

## §1. THE GROUP.

1. Any translation $T$ of the plane may be written

$$(T) \qquad \begin{aligned} x' &= x+h \\ y' &= y+k, \end{aligned}$$

where $h$ and $k$ are the components, in the direction of the coördinate axes, of the vector corresponding to the translation. It is obvious that the totality of translations form a group; for the combination of any two, say $T_1$ and $T_2$ whose vector components are $h_1$, $k_1$ and $h_2$, $k_2$ respectively, gives